Gravitation and Cosmology in Weyl Space-Time

Carlos Romero

Universidade Federal da Paraíba (UFPB)

Table of contents

Introduction

Introduction

- Why study Weyl geometry?
- Why study alternative (modified) theories of gravity?
- Open questions not fully solved by General Relativity
- The problem of the accelerated expansion of the Universe
- Many proposals introduce a scalar field to explain

  Inflation Dark Matter Dark Energy

- What is the nature of this field? Where does it come from?
- The quest for unification of gravity with the other fundamental interactions

  Quantum gravity

Many people believe we now need a new physics. And/or a new geometry...

What is Weyl geometry?

- Weyl geometry is one of the most simple generalizations of Riemannian geometry
- It was formulated by Weyl, in 1918, originally as an attempt to unify gravity and electromagnetism
- This was done by modifying the Riemannian compatibility condition between the metric and the affine connection
- Weyl introduced a new geometric object in the space-time manifold: a 1-form field σ
- If V and W are parallel transported vector fields along a curve C parametrized by λ, then the new compatibility condition is given by

$$\frac{d}{d\lambda}g(V,W) = \sigma(\frac{d}{d\lambda})g(V,W)$$

- As a consequence, if we parallel transport a vector field V along a closed curve C, then the length of V changes according to

$$|V| = |V_0|\, e^{\frac{1}{2}\oint \sigma_\mu dx^\mu}$$

Or, by applying Stokes' theorem,

$$|V| = |V_0|\, e^{\frac{1}{4}\int F_{\mu\nu} dx^\mu \wedge dx^\nu}$$

- The change in the length is regulated by the 2-form $F = d\sigma$ (second curvature or length curvature)
- The length curvature (in coordinates, $F_{\mu\nu} = \partial_\mu \sigma_\nu - \partial_\nu \sigma_\mu$) is invariant under $\sigma \to \overline{\sigma} = \sigma + df$

- For the compatibility condition to be invariant the metric g has to transform as $g \to \overline{g} = e^f g \quad$ (gauge transformation)

- Endowing the space-time manifold with these two ingredients (g and σ) Weyl realized he could obtain a unified theory of gravity and electromagnetism.

- The new theory should be invariant with respect to gauge transformations (*principle of gauge invariance*)

- The generalization of the Riemannian compatibility condition leads to a modification of the Levi Civita's theorem

  In local coordinates, for torsionless manifolds, we have

$$\Gamma^{\mu}{}_{\alpha\beta} = \left\{ {\mu \atop \alpha\beta} \right\} - \frac{1}{2}(\delta^{\mu}_{\alpha}\sigma_{\beta} + \delta^{\mu}_{\beta}\sigma_{\alpha} - g_{\alpha\beta}\sigma^{\mu})$$

- If $F = 0$ (null second curvature) there is no electromagnetic field. But, from Poincaré's lemma, $\sigma = d\phi$.

  In this case, we are left with a scalar field ϕ, a purely geometrical object. The manifold which has this particular version of Weyl geometry is known as **Weyl Integrable Space-Time**, abreviated as **WIST.**

- Weyl transformations now are written as $\overline{g} = e^f g$ and $\overline{\phi} = \phi + f$ and the compatibility condition reads

$$\frac{d}{d\lambda} g(V, W) = d\phi(\frac{d}{d\lambda}) g(V, W)$$

- By choosing $f = -\phi$, we get $\overline{\phi} = 0$, and

$$\frac{d}{d\lambda}\overline{g}(V, W) = 0$$

Thus, in this gauge we recover the Riemannian compatibility condition. We call this the **Riemannian gauge** or the **Riemannian frame.**

In this frame, the affine connection coefficients are identical to the Christoffel symbols taken with respect to $\overline{g} = e^{-\phi} g$

This implies that the curvature tensor, the Ricci tensor and the Ricci scalar are all defined in Riemannian terms

Also, the affine geodesics coincide with the metric geodesics in this frame

The first approach

The first approach

- In 1969, Novello started a research programme on **Weyl Integrable Space-Time (WIST)**

  The first step in this direction was a paper which investigated Dirac spinors in WIST (*M. Novello,Nuovo Cimento A* **94**, 954 (1969)

- In 1983, the first scalar-tensor theory of gravity which assumed that the space-time geometric structure was that of WIST was put forward by Novello and Heintzmann (NH) (*Phys. Lett. A* **98**, *10 (1983)*)

- The action of NH-theory is given by

$$S = \int d^4x\sqrt{-g}[R + \omega\Box\phi] + S_m$$

  where R stands for the Ricci scalar defined with respect to Weyl affine connection, ω is a dimensionless parameter, and S_m is the action corresponding to the matter fields

- In 1992, the theory was applied to obtain a non-singular and bouncing cosmological model in which the scalar field was responsible for an inflationary era (*M. Novello, L.A.R. Oliveira, J.M. Salim, E. Elbaz, Int. J. Mod. Phys. D* **1**, *(1992)*)

- Cosmological models with matter (perfect fluid) were investigated later, in which a general principle to prescribe the interaction of the geometric scalar field ϕ with matter and other physical fields was devised (*J.M. Salim and S.L. Sautú, Class. Quant. Grav.* **13**, *363 (1996)*)

- Non-singular models in NH-theory were also obtained by adding a self-interaction potential of the scalar field to the gravitational sector of the action (*H.P .Oliveira, J.M. Salim, S.L. Sautú, Class. Quant. Grav.***14***,2833 (1997))*

The second approach

The second approach

- A second approach to geometrical scalar-tensor theory appeared more recently (2014), and started from the action of Brans-Dicke-Jordan gravity (*T.S. Almeida, M.L. Pucheu, C. Romero, J.B. Formiga, Phys. Rev. D* **89**, *64047 (2014))*

- The starting point of this second proposal is the action

$$S = \int d^4x\sqrt{-g}\left\{e^{-\phi}\left[R + \omega(\phi)g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right] - V(\phi)\right\} + S_m(g, \Psi),$$

  and to distinguish from the former we shall refer to this approach as the **Weyl geometrical scalar-tensor theory (WGST)**

- In the above action the scalar-field ϕ is regarded as a purely geometrical field, whose meaning becomes clear only after a Palatini variation of the action has been carried out.

- The variation of the action with respect to the affine connection then leads to

$$\nabla_\alpha g_{\mu\nu} = \phi_{,\alpha} g_{\mu\nu}$$

This is the Weyl compatibility condition for a Weyl Integrable Space-Time.

- The variation with respect to g and ϕ leads to the general field equations

$$G_{\mu\nu} = \omega(\phi)\left(\frac{\phi_{,\alpha}\phi^{,\alpha}}{2} g_{\mu\nu} - \phi_{,\mu}\phi_{,\nu}\right) - \frac{1}{2} e^{\phi} g_{\mu\nu} V(\phi) - \kappa T_{\mu\nu}$$

$$\Box\phi = -\left(1 + \frac{1}{2\omega}\frac{d\omega}{d\phi}\right)\phi_{,\mu}\phi^{,\mu} - \frac{e^{\phi}}{\omega}\left(\frac{1}{2}\frac{dV}{d\phi} + V\right)$$

- It is to be noted that $\gamma = e^{-\phi} g$ defines an invariant metric with respect to Weyl transformations.

  This fact allows us to define the energy-momentum tensor in an invariant way:

$$\delta S_m = \int d^4x \sqrt{-g} T_{\mu\nu}(\delta\gamma^{\mu\nu})$$

  It also allows to define many invariant scalars formed only with γ ($\gamma^{\mu\nu} R_{\mu\nu}, R^{\alpha\beta} R_{\alpha\beta}, R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta}$,etc)

  Invariant proper time is defined as

$$\tau = \int_{\lambda a}^{\lambda_b} d^4x \sqrt{-g} e^{-2\phi} \sqrt{g(\frac{d}{d\lambda}, \frac{d}{d\lambda})} d\lambda$$

- In the Riemann frame the action takes the form

$$S = \int d^4x\sqrt{-\gamma}(\overline{R} + \omega\gamma^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - e^{2\phi}V(\phi)) + S_m(\gamma,\psi)$$

- The field equations in this frame are

$$\bar{G}_{\mu\nu} = \omega(\phi)\left(\frac{\phi_{,\alpha}\phi^{,\alpha}}{2}\gamma_{\mu\nu} - \phi_{,\mu}\phi_{,\nu}\right) - \frac{e^{2\phi}}{2}\gamma_{\mu\nu}V(\phi) - \kappa T_{\mu\nu}(\gamma)$$

$$\bar{\Box}\phi = -\frac{1}{2\omega}\frac{d\omega}{d\phi}\phi_{,\alpha}\phi^{,\alpha} - \frac{e^{2\phi}}{\omega}\left(V + \frac{1}{2}\frac{dV}{d\phi}\right)$$

where $\bar{G}_{\mu\nu}$ is the Einstein tensor calculated from the Levi-Civita connection with respect to the metric $\gamma_{\mu\nu}$

- The predictions of Weyl geometrical scalar-tensor theory for gravitational tests within the solar system agree with experiments

- Exterior solutions for spherically symmetric and static matter configurations are given by

$$d\bar{s}^2 = (1-\frac{r_0}{r})^{\frac{M}{\eta}}dt^2-(1-\frac{r_0}{r})^{-\frac{M}{\eta}}dr^2-r^2(1-\frac{r_0}{r})^{1-\frac{M}{\eta}}(d\theta^2+\sin^2\theta d\psi^2)$$

$$\varphi = \frac{\Sigma}{\eta\sqrt{2}}\ln|1-\frac{r_0}{r}|$$

According to the value assigned to the parameter ω, three type of configurations are possible

BLACK HOLE NAKED SINGULARITY WORMHOLE

Cosmology

Cosmology

Recently some cosmological models in Weyl scalar-tensor theory for different choices of the potential $V(\phi)$ and $\omega(\phi) = \omega = const.$ in the absence of matter were obtained (*M.L. Pucheu, F.A.P. Alves Junior, A.B. Barreto, Phys. Rev. D* **94**, 64010 (2016))

$$S = \int d^4x\sqrt{-g}e^{-\phi}(R + \omega\phi^{,\alpha}\phi_{,\alpha} - V(\phi))$$

Considering a homogeneous and isotropic model with flat spatial sections, whose line element is written in the form

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right)$$

the field equations are

$$3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2} = \frac{\omega}{2}\dot{\phi}^2 + \frac{e^{2\phi}}{2}V(\phi)$$

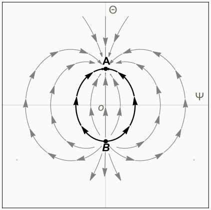
$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -\frac{\omega}{2}\dot{\phi}^2 + \frac{e^{2\phi}}{2}V(\phi)$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = -\frac{e^{2\phi}}{\omega}\left(V(\phi) + \frac{1}{2}\frac{dV}{d\phi}\right)$$

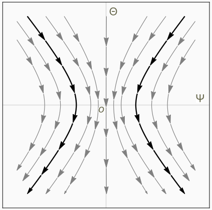
Exact solutions for different choices of the potential with the respective phase portraits were obtained:

- cosmological constant - $V(\phi) = \Lambda e^{-2\phi} V(\phi)$
- exponential potential - $V(\phi) = V_0 e^{-(\lambda+2)\phi}$
- massive potential - $V(\phi) = e^{-2\phi}(m^2\phi^2 + \Lambda)$
- quartic potential - $V(\phi) = 2\lambda(\phi^2 - \beta)^2 e^{-2\phi}$

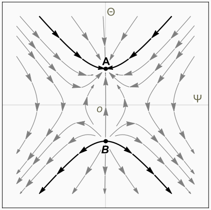
Phase Portrait of the case $V(\phi) = \Lambda e^{-2\phi}$ $(\psi = \dot{\phi})$



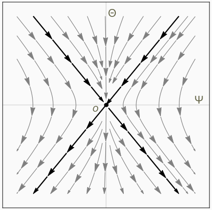
(a) $\omega < 0$ and $\Lambda > 0$



(b) $\omega > 0$ and $\Lambda < 0$

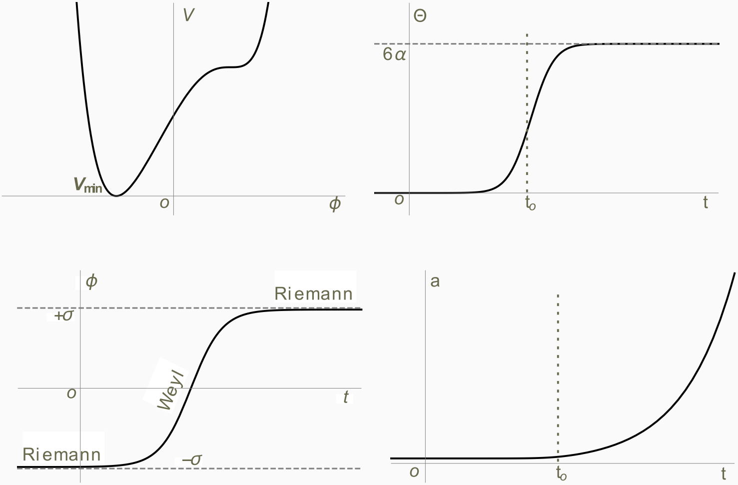


(c) $\omega > 0$ and $\Lambda > 0$



(d) $\omega > 0$ and $\Lambda = 0$

Geometric phase transition



Further developments

Further developments

- **Hawking-Penrose singularity theorems, Raychaudhuri equation and the ADM formulation were extended to WIST**
  (*I. P. Lobo, A. B. Barreto, C. Romero, Eur. Phys. J. C* **75** *, 448 (2015)*, *A. B. Barreto, T. S. Almeida, C. Romero, II COSMOSUR, AIP Conf. Proc. 1647 89-93 (2015)*)

- **(2+1)-Dimensional Gravity in Weyl Integrable Spacetime**

  Main results, unlike general relativity, are

  - the theory has a Newtonian limit for any dimension $n \geq 3$
  - in three dimensions the congruence of world lines of particles of a pressureless fluid has a non-vanishing geodesic deviation
  - for certain values of the parameter ω, space-time has a naked singularity at the center of the matter distribution.

  All these new results are a direct consequence of the Weyl space-time geometry

(*J. E. Madriz Aguilar, C. Romero, J. B. Fonseca-Neto,T. S. Almeida, J. B. Formiga, Class. Quant. Grav.* **32**, *21500 (2015)*)

- **Relativity - its formulation in Weyl Integrable Space-Time** (*C. Romero, J. B. Fonseca-Neto, M. L. Pucheu, Class.Quant.Grav. 29, 155015 (2012)*)

  The n-dimensional invariant action is given by

$$S = \int d^n x \sqrt{-g} e^{(1-\frac{n}{2})\phi} (R + 2\Lambda e^{-\phi} + \kappa e^{-\phi})$$

  Similar results were obtained by F. P. Poulis and J. M. Salim in , *Int. J .Mod. Phys. Conf. Ser. 3, 87 (2011)*

- **Conformally flat space-time and Weyl geometry**

  A new picture arises when a Riemannian spacetime is taken by means of geometrical gauge transformations into a Minkowskian flat spacetime. We find out that in the Weyl frame gravity is described by a scalar field. We give some examples of how conformally flat spacetime configurations look when viewed from the standpoint of a Weyl frame. We show that in the non-relativistic and weak field regime the Weyl scalar field may be identified with the Newtoniangravitational potential. We suggest an equation for the scalar field by varying the Einstein-Hilbert action restricted to the class of conformally-flat spacetimes. We revisit Einstein and Fokker's interpretation of Nordström scalar gravity theory and draw an analogy between this approach and the Weyl gauge formalism. We briefly take a look at two-dimensional gravity as viewed in the Weyl frame and address the question of quantizing a conformally flat spacetime by going to the Weyl frame
  (*C. Romero, J. B. Fonseca-Neto, M. L. Pucheu, Found.Phys. 42, 224 (2012)*)

- **The embedding of Weyl manifolds**

  Extension of different versions of the Campbell-Magaard theorem formulated in Riemannian geometry to Weyl geometry.
  (These embedding theorems have relevance for higher-dimensional theories)

  (*R. Avalos, F. Dahia, C. Romero, J. Math. Phys.* **58**, *012502 (2017)*)

- **Quantum Cosmology in Weyl Integral Space-Time**

  Quantum cosmological models in an n-dimensional anisotropic universe in the presence of a massless scalar field. Our basic inspiration comes from Chodos and Detweiler's classical model which predicts an interesting behaviour of the extra dimension, shrinking down as time goes by. We work in the framework of a recent geometrical scalar-tensor theory of gravity. Classically, we obtain two distinct type of solutions. One of them has an initial singularity while the other represents a static universe considered as a whole. By using the canonical approach to quantum cosmology, we investigate how quantum effects could have had an influence in the past history of these universes
  (A.F.P. Alves, M. L. Pucheu, A. B. Barreto and C. Romero, ArXiv: 1611.03812 [gr-qc])

Weyl non-integrable space-time

Weyl's original unified theory

The invariant action (geometric sector) is

$$S = \int d^4x\sqrt{-g}[R^2 + \omega F_{\mu\nu}F^{\mu\nu}]$$

Variations with respect to σ_μ and $g_{\mu\nu}$ lead to the field equations

$$\partial_\nu\left(\sqrt{-g}F^{\mu\nu}\right) = \frac{3}{2\omega}g^{\mu\nu}\left(R\sigma_\nu + \partial_\nu R\right)$$

$$R(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R) = \omega T_{\mu\nu} - D_{(\mu\nu)}$$

where $T_{\mu\nu} = F_{\mu\alpha}F^{\alpha}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$ and

$$D_{(\mu\nu)} = R_{\mu;\nu} + \frac{1}{2}R(\sigma_{\mu;\nu} + \sigma_{\mu;\nu}) + R\sigma_\mu\sigma_\nu + R\sigma_\nu\sigma_\mu + R_{,\mu}\sigma_\nu + R_{,\nu}\sigma_\mu$$

The equations have a simpler form in the gauge $R = \Lambda$ (*Weyl gauge*)

$$\partial_\nu \left(\sqrt{-g} F^{\mu\nu}\right) = \frac{3\Lambda}{2\omega}\sigma^\mu$$

$$\widetilde{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\widetilde{R} = \frac{\omega}{\Lambda}T_{\mu\nu} - \frac{3}{2}(\sigma_\mu\sigma_\nu - \frac{1}{2}g_{\mu\nu}\sigma_\alpha\sigma^\alpha) - \frac{\Lambda}{4}$$

in which $\widetilde{R}_{\mu\nu}, \widetilde{R}$ are expressed in Riemannian terms

When applied to the case of a static and spherically symmetric uncharged matter distribution Weyl's theory correctly predicts the perihelion precession of Mercury as well as the gravitation deflection of light by a massive body

In fact, this is a consequence of the fact that all vacuum solutions of Einstein's equations (including the Schwarzschild solution) satisfy Weyl field equations when we set $\sigma = 0$

The problem of time and Einstein's objection

Einstein reasoned that in a space-time ruled by Weyl geometry the existence of sharp spectral lines in the presence of an electromagnetic field would not be possible since atomic clocks would depend on their past history. He based his argument essentially in two hypotheses:

- (H1) The proper time $\triangle\tau$ measured by a clock travelling along a curve $\alpha = \alpha(\lambda)$ is given as in general relativity, namely, by the (Riemannian) prescription

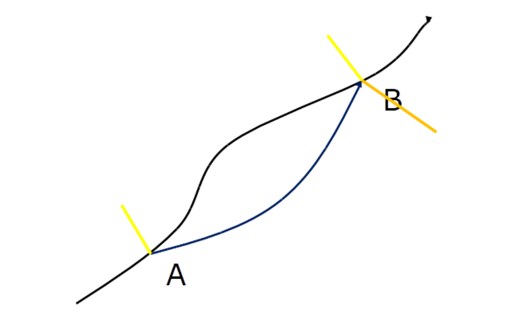
$$\triangle\tau = \int [g(V,V)]^{\frac{1}{2}}\, d\lambda = \int [g_{\mu\nu}V^{\mu}V^{\nu}]^{\frac{1}{2}}\, d\lambda,$$

  where V denotes the vector tangent to the clock's world line. This supposition is known as the *clock hypothesis* and clearly assumes that the proper time only depends on the instantaneous speed of the clock and on the metric field

- (H2) The fundamental clock rate (tic tac) of clocks (in particular, atomic clocks) is to be associated with the (Riemannian) length $L = \sqrt{g(\Upsilon, \Upsilon)}$ of a certain vector Υ. As a clock moves in space-time Υ is parallel transported along its worldline from a point P_0 to a point P, hence $L = L_0 e^{\frac{1}{2}\int \sigma_\alpha dx^\alpha}$, L_0 and L indicating the duration of the tic tac of the clock at P_0 and P, respectively

Second clock effect

Atomic clocks would depend on the location and past history of the atoms. This would contradict the existence of the sharp and well defined spectral lines. (Einstein (1918))



Clock tic tac may change!

Let us now have a look into these two assumptions. We start with the first hypothesis (H1). First of all, we would expect that, to be consistent with the *Principle of Gauge Invariance*, proper time, as a physically-relevant quantity, should be gauge invariant.

It turns out, however, that there is no such invariant notion of proper time in Weyl's theory. In addition to that, the adoption of the general relativistic clock hypothesis here does not seem to be plausible, since $\triangle\tau$, as defined above, takes into account only part of the geometry, namely, the metric field, and completely ignores the other geometric field, i.e., the gauge field σ_μ.

In the second hypothesis (H2), gauge invariance again is violated: the concept of clock rate is not modelled as a gauge-invariant physical quantity and, again, the Weyl geometrical field plays no role in its determination.

Without the problem of time being solved Weyl unified field theory is incomplete

We do not know if it is possible to "complete" the elegant and profound theory developed by Weyl almost a century ago

Some years ago V. Perlick developed a concept of time that leads to a well-defined and physically sensible definition of proper time in a general Weyl space-time (*V. Perlick, Gen. Rel. Grav.* **19***:1059-1073 (1987)*)

However, it has been shown recently that Perlick's time still leads to the prediction of the second clock effect (*R. Avalos, F. Dahia, and C. Romero, (2016), arXiv:1611.10198 [gr-qc]*)

Does the second clock effect exist?

"**Weyl geometrical theory contains a suggestive formalism and may still have the germs of a future fruitful theory** "

(*Ronald Adler, Maurice Bazin, Menahem Schiffer, Introduction to General Relativity, 1965)*

Thank you