GLT Structures, spectral approximation of PDEs, symbol, and Fast Solvers

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From continuous to discrete

A continuous infinite-dimensional problem (PDEs, IEs etc) is transformed, via a suitable numerical approximation, into a linear (nonlinear) system of algebraic equations

- Structure inherited from the continuous counterpart
- Large dimensions (e.g. 10^p, $p \geq 10$)
- Spectral features described via a proper Symbol

Goal: solving the resulting linear system by **Optimal Methods** (operation count to obtain the solution of the same order of the matrix-vector multiplication)
Goal: understanding the spectral properties of the resulting matrices (Weyl formulas:**from discrete to continuous**; information for Engineers)

From continuous to discrete

Linear PDE/IE $Lu = g$

$\Downarrow$

Linear Numerical Method $L_n \mathbf{u}_n = \mathbf{g}_n$

- $\dim(L_n) \to \infty$ as $n \to \infty$
- $\{L_n\}$ has an asymptotic spectral distribution described by a spectral/sv symbol

GLT sequences = a tool for computing spectral/sv symbols

GLT sequences = a tool for designing fast numer. methods

$\{L_n\}$ is usually a GLT sequence

In the discrete case

- Large dimensions imply that direct solvers (Gaussian Elimination etc.) have to be avoided
- Iterative solvers: A) operation count per iteration of the same order of the matrix-vector multiplication B) the method is Optimal if the number of iterations $\leq c(\epsilon)$, with ϵ desired precision.

Requirement B) depends on the spectrum of the involved matrices: it depends especially on the possibility of approximating the coefficient matrix in the ill-conditioned subspaces (i.e. associated to the eigenvectors with small eigenvalues).

$$\Downarrow$$

For large classes of matrices coming from continuous problems, **the knowledge of the spectrum** is often compactly represented in a function, called the **symbol**.

The GLT components I: Toeplitz sequences

Let $f \in L^1([-\pi, \pi])$ with Fourier coefficients

$$f_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \mathrm{e}^{-\mathrm{i}j\theta} d\theta, \qquad j \in \mathbb{Z}$$

$$T_n(f) = \begin{pmatrix} f_0 & f_{-1} & \cdots & \cdots & f_{-(n-1)} \\ f_1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & f_{-1} \\ f_{n-1} & \cdots & \cdots & f_1 & f_0 \end{pmatrix}$$

$$T_n(2 - 2\cos(\theta)) = \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}$$

Its eigenvalues are a sampling of $2 - 2\cos(\theta)$.

The GLT components II: diagonal sampling matrices

Let $a : [0,1] \to \mathbb{C}$

$$D_n(a) = \begin{pmatrix} a(\frac{1}{n}) & & & \\ & a(\frac{2}{n}) & & \\ & & \ddots & \\ & & & a(1) \end{pmatrix}$$

The eigenvalues of $D_n(a)$ are clearly the samplings of $a(x)$.

The GLT algebra: Toeplitz + Diagonal

GLT sequences = The algebra of matrix sequences containing $\{D_n(a)\}$, a Riemann integrable, $\{T_n(f)\}$, f Lebesgue integrable, $\{X_n\}$ zero distributed sequences.

As an example (not academical!)

- $A_n = D_n(a_1)T_n(f_1) + T_n(f_2)X_n + Y_n$, with $\{X_n\}$, $\{Y_n\}$ zero distributed sequences
- $\{A_n\}$ has singular values approximated by an equispaced sampling of $|\psi(x,\theta)|$, $\psi(x,\theta) = a_1(x)f_1(\theta)$
- If $\{A_n\}$ is quasi-Hermitian, then $\{A_n\}$ has eigenvalues approximated by an equispaced sampling of $\psi(x,\theta)$

The GLT idea: Toeplitz + Diagonal

$\mathcal{L}_a(u) = -\left(a(x)u^{'}\right)^{'}$ [rod with variable section].

$$K_n = \begin{pmatrix} d_1 & -a_2 & & \\ -a_2 & \ddots & \ddots & \\ & \ddots & \ddots & -a_n \\ & & -a_n & d_n \end{pmatrix},$$

$$T_n(2-2\cos(\theta)),\ D_n(a) = \mathrm{diag}(a(jh)), \quad h = \frac{1}{n+1}.$$

Then

$$\begin{aligned} K_n &= D_n(a)T_n(2-2\cos(\theta)) + E_n, \quad \|E_n\| \to 0, \\ \psi(x,\theta) &= a(x)(2-2\cos(\theta)). \end{aligned}$$

The eigenvalues of K_n are a sampling of $\psi(x,\theta)$: **this is a GLT result**.

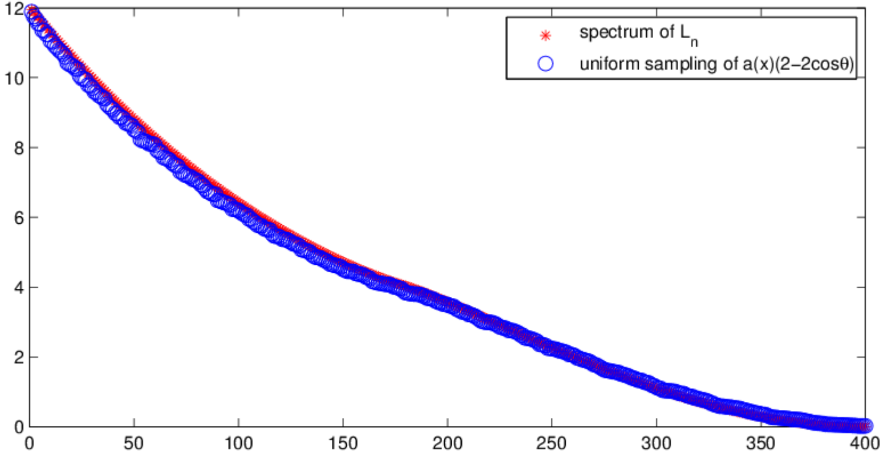


Figure: $a(x) = 2 + \cos(3x)$, $n = 400$. The error on each eigenvalue is of order n^{-1}: can we do better?

Which messages? Very short answers

a.1) GLT sequences are a subspace of sequences of matrices $\{A_n\}$, A_n of size d_n ($d_k < d_{k+1}$)

a.2) Each GLT sequence $\{A_n\}$ has a symbol ψ

a.3) The singular values of a GLT sequence with symbol ψ are approximately described by equispaced sampling of $|\psi|$

a.4) The spectrum of a (quasi Hermitian) GLT sequence with symbol ψ are approximately described by equispaced sampling of ψ

a.5) The GLT sequences are stable under elementary operations and the symbol is obtained via the same elementary operations

The GLT glasses: a variable-coefficient-operative version of the Local Fourier Analysis

b.1) Local methods (including FDs, FEs, IgA, FVs, VEMs) for approximating PDEs, IEs lead to GLT sequences, possibly after proper permutations

b.2) No limitations on variable coefficients and on domains (grids should have some structure at least asymptotically)

b.3) Information on the symbol leads to information on ill-conditioning, on the size of the ill-conditioned subspaces, on the nature of the ill-conditioned subspaces (low frequencies, high frequencies etc)

The GLT glasses.....

The GLT glasses: a variable-coefficient-operative version of the Local Fourier Analysis

The GLT glasses.....

c.1) We exploit the symbol for understanding the reason of difficulties of known techniques, w.r.t. finess parameters, problem parameters, approximation parameters

c.2) We exploit the symbol for designing new iterative solvers, new preconditioners or smoothers or prolongation operators, aiming at optimality and robustness.

Main items

Symbol for matrix sequences

1. Toeplitz, Diagonal structures and symbol
2. Approximation of Differential Operators
3. The GLT algebra and the notion of symbol

Examples + (preconditioning, multigrid)

4. FEM of degree p in d dimensions
5. Approximation Q2Q1 of the Linear Elasticity
6. IgA of degree p in d dimensions
7. FDEs and symbol approach

Collaborators

Ahmad, Al Aidarous, Barbarino, Beckermann, Benedusi, Bertaccini, Bianchi, Böttcher, R. Chan, Di Benedetto, Donatelli, Dorostkar, Dumbser, Durastante, Ekstrom, Fiorentino, Franck, Furci, Garoni, Golub, Golinskii, Krause, Hughes, Kuijlaars, Manni, Mazza, Molteni, Neytcheva, Pelosi, Pennati, Ratnani, Reali, Semplice, Sesana, Sonnendrücker, Speleers, Tablino Possio, Tavelli, Tilli, Tyrtyshnikov, Zuazua.

- In blue consolidated collaborations on the themes of the talk;
- In green just started collaborations (with the goal of variable-coeff. vector PDEs).

$$\Downarrow$$

Elasticity, Navier-Stokes, MHD,...

Spectral Distribution: the qualitative idea

- $M_m(\mathbb{C})$ complex matrices of order m,
- $\{A_n\}$, $A_n \in M_{d_n}(\mathbb{C})$, $d_n < d_{n+1}$,
- ψ measurable on $D \subset \mathbb{R}^g$, $g \geq 1$,
- ψ being $M_s(\mathbb{C})$-valued, $s \geq 1$,
- $0 < \mu\{D\} < \infty$, $\mu\{\cdot\}$ can be the Lebesgue measure,

$$\{A_n\} \sim_\lambda (\psi, D).$$

Informal meaning: $s = 1$. If ψ is continuous, then a suitable ordering of the eigenvalues $\{\lambda_j(A_n)\}$, in correspondence with a equispaced gridding on D, reconstructs approximately the surface $t \to \psi(t)$.
Informal meaning: $s > 1$. If ψ is continuous, then a suitable ordering of the eigenvalues $\{\lambda_j(A_n)\}$, in correspondence with a equispaced gridding on D, reconstructs approximately s surfaces, $t \to \lambda_j(\psi(t))$, $j = 1, \ldots, s$.

Spectral Distribution: the definition

$F \in C_0$ (continuous with compact support):

$$\Sigma_\lambda(F, A_n) = \frac{1}{d_n} \sum_{j=1}^{d_n} F[\lambda_j(A_n)].$$

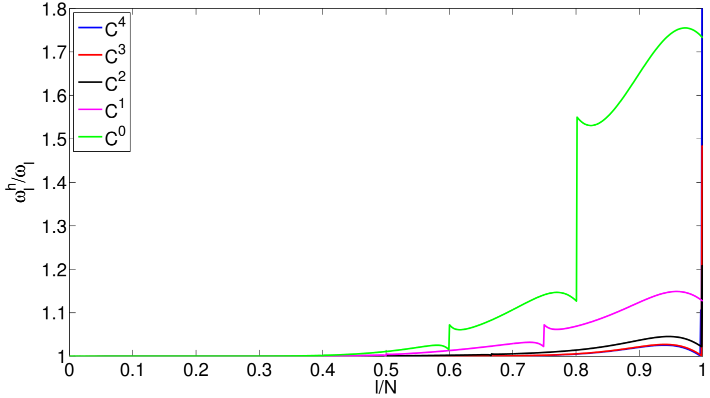
Definition

We write $\{A_n\} \sim_\lambda (\psi, D)$ if $\forall F \in C_0$

$$\lim_{n\to\infty} \Sigma_\lambda(F, A_n) = \frac{1}{s\mu\{D\}} \int_D \mathrm{trace}\left(F\left(\psi(t)\right)\right)\, dt.$$

Moreover, we write $\{A_n\} \sim_\sigma (\psi, D)$ replacing $\lambda_j(A_n)$ by $\sigma_j(A_n)$ (singular values) in $\Sigma_\sigma(F, A_n)$ in place of $\Sigma_\lambda(F, A_n)$ and replacing $\psi(t)$ by $|\psi(t)|$ in the integral. If $s > 1$ then $|\psi(t)| = (\psi^*(t)\psi(t))^{1/2}$.

Comparison IgA-FEM (and furthermore the case of intermediate regularity): C^0 →FEM→ $s = p^d$, C^{p-1} →IgA→ $s = 1$, C^k →interm. regularity → $s = (p-k)^d$ (figure by A. Reali)



Toeplitz sequences generated by a symbol: $\{T_n(f)\} \sim_\lambda (f, I_d)$ if $f = f^*$

- s, d positive integers, $\mathbf{i}^2 = -1$;
- $f \in L^1(I_d, M_s(\mathbb{C}))$, $I_d = (-\pi, \pi)^d$, $j \in \mathbb{Z}^d$;
- $f_j = \frac{1}{(2\pi)^d} \int_{I_d} f(s) e^{-\mathbf{i}js} \, ds$, $f_j \in M_s(\mathbb{C})$.

For $d = 1$ the matrix $T_n(f)$ has size ns:

$$T_n(f) = \begin{pmatrix} f_0 & f_{-1} & \cdots & f_{1-n} \\ f_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & f_{-1} \\ f_{n-1} & \cdots & f_1 & f_0 \end{pmatrix}.$$

For $d > 1$ we have a recoursive formula.

Toeplitz sequences generated by a symbol: $\{T_n(f)\} \sim_\lambda (f, I_d)$ if $f = f^*$

For $d > 1$, the d-level Toeplitz matrix $T_n(f)$ has order Ns, $N = \prod n_j$, $n = (n_1, \ldots, n_d)$, and takes the form

$$T_n(f) = \begin{pmatrix} T_0 & T_{-1} & \cdots & T_{1-n_1} \\ T_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & T_{-1} \\ T_{n_1-1} & \cdots & T_1 & T_0 \end{pmatrix},$$

T_j being $(d-1)$-level Toeplitz matrix. If $\otimes$ denotes the Kronecker product

$$T_n(f) = \sum_{|j| \leq n-1} J_n^{[j]}, \qquad J_n^{[j]} = J_{n_1}^{j_1} \otimes \cdots \otimes J_{n_d}^{j_d} \otimes f_j,$$

with $(J_m^r)_{s,t} = 1$ if $s - t = r$ and 0 otherwise.

FEM: of degree p on a d dimensional domain

We consider the Laplacian over $[0,1]^d$ and we denote by $A_n^{(p)}$ the degree p FEM matrix on quadrilaterals.

- There exists a permutation matrix Π such that

$$\Pi A_n^{(p)} \Pi^T \approx T_n(f);$$

- f is defined over $I_d = (-\pi, \pi)^d$ and Hermitian matrix-valued with size p^d (any comment is redundant!);
- hence, the eigs of $A_n^{(p)}$ are divided into p^d branches (of the same cardinality), each of them represented by a different real-valued eigenvalue of f: $\lambda_1(f) \leq \ldots \leq \lambda_{p^d}$;
- the spreading of the spectrum, measured by the ratio

$$\frac{\max(\lambda_{p^d})}{\max(\lambda_1)},$$

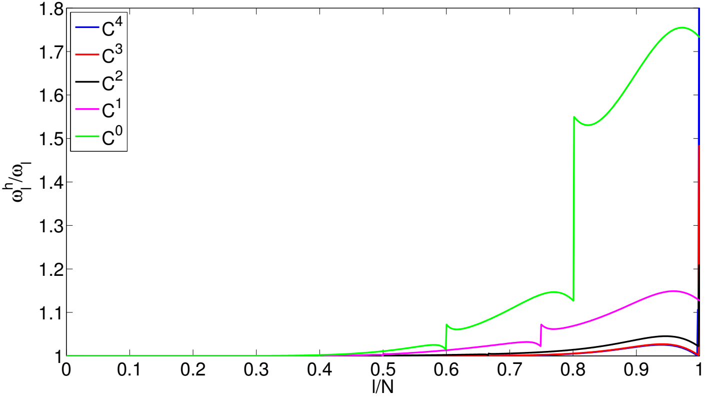
  depends on the choice of the basis (Lagrange, integrated Legendre, Bernstein etc); not that of $[M_n^{(p)}]^{-1} A_n^{(p)}$.

IgA: of degree p on a d dimensional domain

We consider the Laplacian over $[0,1]^d$ and we denote by $A_n^{(p)}$ the spline-degree p IgA matrix.

- It holds $A_n^{[p]} \approx T_n(f)$ so that $\{A_n^{(p)}\} \sim_\lambda (f, I_d)$, $I_d = (-\pi,\pi)^d$;
- f is defined over $I_d = (-\pi,\pi)^d$, is scalar-valued, nonnegative with a unique zero at zero (as in the FD case: it is somehow the revenge of the smoothness);
- the function f tends exponentially to zero as p in every point of the type $\theta = (\theta_1, \ldots, \theta_d)$ for which $\theta_j = \pi$ for some j;
- the latter property induces a bad conditioning in the high frequency subspace, growing exponentially with p and which is not expected for a differential problem: the knowledge of the symbol is an essential guide for finding the right preconditioner.

Comparison IgA-FEM (and furthermore the case of intermediate regularity): **the picture has a clear interpretation as the revenge of the smoothness**



IgA, degree p, d dimensions: spectral distribution and classical multigrid

Theorem $n^{d-2}A_n^{[p]} \approx T_n(f_p)$ **and hence** $\{n^{d-2}A_n^{[p]}\}_n \sim_\lambda (f_p, I_d)$

f_p has the expected zero of order 2 at zero, positive elsewhere but it collapses to zero exponentially with p at the boundaries of $I_d = (-\pi, \pi)^d$.

n	$p=1$	$p=3$	$p=5$
16	0.16	0.64	0.96
28	0.17	0.64	0.96
40	0.18	0.64	0.96
52	0.18	0.65	0.96
n	$p=2$	$p=4$	$p=6$
17	0.27	0.88	0.99
29	0.27	0.88	0.99
41	0.29	0.88	0.99
53	0.30	0.88	0.99

Table: spectral radius: standard twogrid, 2D, relaxed GS as smoother

$\approx$ denotes equality up to matrix-sequences with zero symbol.

IgA, degree p, d dimensions: structured PCG/PGMRES and multigrid (const. coeff.... but the technique is equally effective for var. coeff. and singular mappings)

We consider the system $nA_n^{[p]}\mathbf{u} = \mathbf{b}$ coming from the IgA approximation of

$$\begin{cases} -\Delta u = 1 & \text{in } (0,1)^3 \\ u = 0 & \text{on } \partial(0,1)^3 \end{cases}$$

For the solution: V-cycle and W-cycle multigrid

n	$p=1$		n	$p=3$		n	$p=5$	
16	10	7	14	7	6	12	8	8
32	11	7	30	8	6	28	8	7
64	12	7	62	9	6	60	9	6

n	$p=2$		n	$p=4$		n	$p=6$	
15	9	8	13	7	6	11	9	9
31	8	7	29	8	6	27	8	6
63	9	7	61	9	6	59	10	6

Table: number of iterations: 3D with structured PCG/PGMRES

IgA, degree p, d dimensions: structured PCG/PGMRES and multigrid

We consider the system $nA_n^{[p]}\mathbf{u} = \mathbf{b}$ coming from the IgA approximation of

$$\begin{cases} -\Delta u = 1 & \text{in } (0,1)^3 \\ u = 0 & \text{on } \partial(0,1)^3 \end{cases}$$

Only one (!) new ingredient in our fast V-cycle

- Standard restriction and prolongation operator;
- Standard smoother (GS) at coarse grids;
- V-cycle with PCG/PGMRES as smoother only at the finest grid;
- Preconditioner chosen by using the information contained in the symbol;
- The preconditioner has a very cheap tensor-banded structure (Tani's talk)

Variable coefficients: symbol of the IgA matrix-sequences associated to a full elliptic Pb (a GLT sequence)

Full elliptic problem:

$$\begin{cases} -\nabla \cdot K \nabla u + \boldsymbol{\beta} \cdot \nabla u + \gamma u = f & \text{on } \Omega \subset \mathbb{R}^d \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

IgA approximation: take a geometry map $\mathbf{G} : [0,1]^d \to \overline{\Omega}$ to transfer the problem from Ω to $[0,1]^d$; on $[0,1]^d$ use again splines of deg. p.

$$\mathcal{A}_n^{[p]} = \text{resulting IgA approximation matrix}$$

Theorem $\{n^{d-2}\mathcal{A}_n^{[p]}\}$ sequence of matrices in the GLT super-algebra

$$\boxed{\{n^{d-2}\mathcal{A}_n^{[p]}\} \sim_\lambda \mathbf{1}\big(|\det(J_{\mathbf{G}}(x_1,\ldots,x_d))| K_{\mathbf{G}}(x_1,\ldots,x_d) \circ H_p(\theta_1,\ldots,\theta_d)\big)\mathbf{1}^T}$$

$$K_{\mathbf{G}} = (J_{\mathbf{G}})^{-1} K(\mathbf{G}) (J_{\mathbf{G}})^{-T}, \quad J_{\mathbf{G}} = \text{Jacobian matrix of } \mathbf{G}$$

$H_p =$ symmetric $d \times d$ matrix whose (i,j) entry represents the 'formula' used to approximate $\partial^2/\partial x_i \partial x_j$

The symbol: Toeplitz and GLT through an example

Minimalistic example

Assume

$$\begin{cases} -(\kappa_0 u')' + v' & = & g_1(x), \\ u' - \rho v & = & g_2(x), \end{cases}$$

Discretize on a square mesh of stepsize h using bilinear FEM basis functions:

$$\mathcal{A} = \begin{bmatrix} K & B^T \\ B & -\rho M \end{bmatrix}$$

The symbol: Toeplitz and GLT through an example

Minimalistic example, cont.: the arising matrices

$$K = \kappa_0 \cdot \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad M = \frac{h^2}{6} \begin{bmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 4 \end{bmatrix},$$

$$B = h \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}.$$

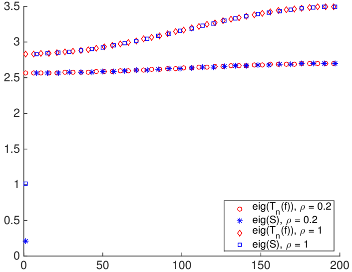
The symbol: Toeplitz and GLT through an example

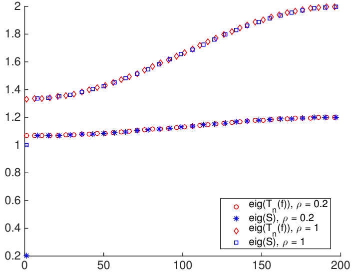
Minimalistic example, cont.: matrices and symbols

$$
\begin{aligned}
K &= \kappa_0\, T_n(2-2\cos(\theta)), & B &= h\, T_n(1-e^{i\theta}), \\
B^T &= h\, T_n(1-e^{-i\theta}), & M &= \tfrac{h^2}{3}\, T_n(2+\cos(\theta)).
\end{aligned}
$$

$$
S=\rho M+B^TK^{-1}B, \quad \left\{\frac{S}{h^2}\right\} \sim_\lambda \left(\frac{\rho}{3}(2+\cos(\theta))+\frac{1}{\kappa_0},(-\pi,\pi)\right).
$$

Below $\kappa_0 = 0.4$ and $\kappa_0 = 1$: **if $\kappa_0 = a(x)$ the formula of the symbol (a bivariate function in (x,θ)) is formally exactly the same!**





A Concrete Example: Q2Q1 approximation of the Linear Elasticity

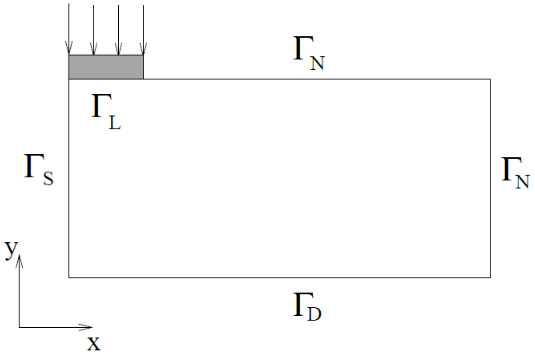
Elasticity - as a part of the ‘rebound’ analysis in the Glacial Isostatic Adjustment (GIA) models, Donation KAW 2013.0341, Knut & Alice Wallenberg Foundation, in collaboration with the Royal Swedish Academy of Sciences

Elasticity Pb: some simplifications

To enable fully incompressible models, i.e. $\lambda = \infty$, we write

$$\begin{aligned} -\nabla \cdot (2\mu\varepsilon(\mathbf{u})) - \nabla(\mathbf{u} \cdot \nabla p_0) - \mu\nabla p &= \mathbf{f} \text{ in } \Omega \\ \mu\nabla \cdot \mathbf{u} - \frac{\mu^2}{\lambda} p &= 0 \text{ in } \Omega \end{aligned}$$

- p_0 is the pre-stress,
- $p = \frac{\lambda}{\mu}\nabla \cdot \mathbf{u}$ is the kinematic pressure.



Q2Q1 for the Elasticity Pb: two-by-two block structure (Stokes, NS, Cahn-Hilliard, PDE constrained opt. etc)

We use the stable pair of spaces Q2-Q1; we obtain a block structure $\mathcal{A}$ with a block factorization

$$\mathcal{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & S_{\mathcal{A}} \end{bmatrix} \begin{bmatrix} I_1 & A_{11}^{-1}A_{12} \\ 0 & I_2 \end{bmatrix},$$

$$S_{\mathcal{A}} = A_{22} - A_{21}A_{11}^{-1}A_{12}.$$

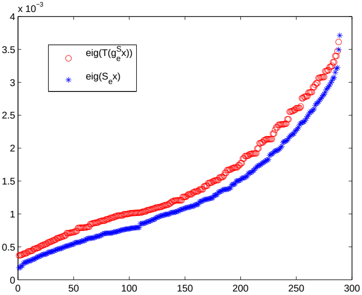
Several possible preconditioners. One option is

$$\mathcal{D} = \begin{bmatrix} D_{11} & 0 \\ A_{21} & S \end{bmatrix},$$

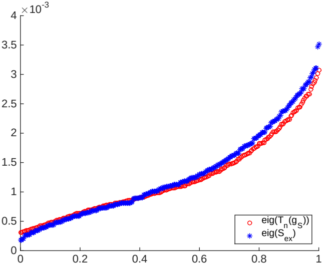
$$D_{11} \approx A_{11}, \quad \text{or} \quad D_{11} \approx A_{11}^{-1} \quad \text{and} \quad S \approx S_{\mathcal{A}}$$

To be computationally efficient, in the majority of the cases, the Schur complement must be approximated.

Approximation of the exact Schur compl $S_{\mathcal{A}}$ (case Q1-Q1): eig(S) in blue vs the GLT symbol $g_S(\theta_1, \theta_2)$ in red



Approximation of the exact Schur compl $S_{\mathcal{A}}$ (case Q2-Q1): eig(S) in blue vs the GLT symbol $g_S(\theta_1, \theta_2)$ in red



Again on Toeplitz sequences generated by a symbol f

- s, d positive integers, $\mathbf{i}^2 = -1$;
- $f \in L^1(I_d, M_s(\mathbb{C}))$, $I_d = (-\pi, \pi)^d$, $j \in \mathbb{Z}^d$;
- $f_j = \frac{1}{(2\pi)^d} \int_{I_d} f(s) e^{-\mathbf{i} js}\, ds$, $f_j \in M_s(\mathbb{C})$.

$$T_n(f) = \begin{pmatrix} f_0 & f_{-1} & \cdots & f_{1-n} \\ f_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & f_{-1} \\ f_{n-1} & \cdots & f_1 & f_0 \end{pmatrix}.$$

We have

- $\{T_n(f)\} \sim_\lambda (f, I_d)$ if $f = f^*$;
- $\{T_n(f)\} \sim_\sigma (f, I_d)$ (no assumption needed).

Sequences of diagonal (uniform) sampling matrices

- Let $a(x)$ be Riemann integrable over $(0,1)$ and let us consider the (uniform) diagonal sampling matrix of size n

$$D_n(a) = \begin{pmatrix} a(h) & & & \\ & a(2h) & & \\ & & \ddots & \\ & & & a(nh) \end{pmatrix}, \quad h = \frac{1}{n+1}.$$

  It is plain to see that

$$\{D_n(a)\} \sim_{\lambda,\sigma} (a, \Omega), \quad \Omega = (0,1).$$

- The result is also immediate for domains $\Omega \in \mathbb{R}^d$ measurable according to Peano-Jordan (and even for matrix-valued symbols).
- Are we satisfied? ... We have to wait a bit ...

Approximation Theory for matrix-sequences I

Definition [Tilli LAA 98,S. LAA 01](... a suggestion by De Giorgi)
For $\{A_n\}$ con $d_n < d_{n+1}$
$\{\{B_{n,m}\}\}_m$, $m \in \mathbb{N}$ is an a.c.s. (approximating class) if

$$A_n = B_{n,m} + R_{n,m} + N_{n,m}, \quad \forall n > n_m, \ \forall m \in \mathbb{N},$$

$$\mathrm{rank}\ R_{n,m} \leq d_n\, c(m), \quad \|N_{n,m}\| \leq \omega(m).$$

The quantities n_m, $c(m)$ and $\omega(m)$ are functions of m and

$$\lim_{m\to\infty} \omega(m) = 0, \quad \lim_{m\to\infty} c(m) = 0.$$

Approximation Theory for matrix-sequences II

Theorem [Tilli LAA 98, S. LAA 01-06]
Assuming the following

- $\{\{B_{n,m}\}\}_m$, $m \in \hat{\mathbb{N}} \subset \mathbb{N}$, $\#\hat{\mathbb{N}} = \infty$ a.c.s. per $\{A_n\}$,
- $\{B_{n,m}\} \sim_\sigma (\psi_m, D)$ ($\{B_{n,m}\} \sim_\lambda (\psi_m, D)$),
- $\psi_m \to_\mu \psi$,

we obtain

$$\{A_n\} \sim_\sigma (\psi, D) \quad (\{A_n\} \sim_\lambda (\psi, D)).$$

In the case of eigenvalues every involved matrix-sequence has to be Hermitian (or the non-Hermitian perturbation has to satisfy a trace norm condition; trace norm = sum of all singular values).

GLT: Generalized Locally Toeplitz [S. et al, 03-15]

We know a lot on spectral features of either Toeplitz or Diagonal matrix-sequences: exploiting these ‘two ingredients’, via the a.c.s., notion we build up a class of matrix-sequences called Generalized Locally Toeplitz (GLT):

- the technique relies on the a.c.s. notion (small rank plus small norm);
- the a.c.s technique is a generalization of that used in Precondtioning from structured matrices (R.Chan etc)
- the small rank plus small norm idea was used by Tyrtyshnikov for proving the Szegö Theorem;
- small rank plus small norm decompositions are a key ingredient for Mosaic Rank, Semiseparable, Tensor Trains, H matrices etc;
- the idea of a.c.s. was formalized by Tilli and S., but there was a 'suggestion' by E. De Giorgi.

GLT: Generalized Locally Toeplitz [S. et al, 03-15]

Toeplitz + Diagonal + a.c.s. notion Generalized Locally Toeplitz (GLT) matrix-sequences:

- for a Riemann integrable over $[0,1]$ and f being $L^1(-\pi,\pi)$, we define $LT_n^m(a,f) = D_m(a) \otimes T_{n/m}(f)$;
- a sequence $\{A_n\}$ is sLT if $\{\{LT_n^m(a,f)\}\}_m$ is an a.c.s. for $\{A_n\}$: in that case $a(x)f(\theta)$, $(x,\theta) \in [0,1] \times (-\pi,\pi)$, is the symbol of the sequence of matrices $\{A_n\}$;
- a sequence $\{A_n\}$ is GLT with respect to the measurable function $\kappa(x,\theta)$ if for every $\epsilon > 0$, there exist $\{A_n^{(j,\epsilon)}\}$ sequences sLT with symbol $a_{(j,\epsilon)}(x)f_{(j,\epsilon)}(\theta)$, $N_\epsilon \geq j \geq 1$ such that
  - $\sum_{j=1}^{N_\epsilon} a_{(j,\epsilon)}(x)f_{(j,\epsilon)}(\theta)$ converges a.e. to $\psi(x,\theta)$;
  - $\{\{\sum_{j=1}^{N_\epsilon} A_n^{(j,\epsilon)}\}\}_{m,\epsilon}$ is a.c.s. for $\{A_n\}$.
- a GLT sequence $\{A_n\}$ GLT with respect to the measurable function $\psi(x,\theta)$ has $\psi(x,\theta)$ as symbol (extensions for $d > 1$).

GLT as super-algebra containing $\{D_n(a)\}$, $\{T_n(f)\}$, a Riemann integrable, f belonging to L^1

- Any linear combination of products (and inverses) involving uniform sampling diagonal matrix-sequences and Toeplitz sequences is GLT and has as symbol the function obtained by the same operations on the symbols...... sequences distributed in the singular value sense as the zero function are GLT with $\psi \equiv 0$.
- Surprisingly enough, we prove formally that any 'reasonable' approximation by local methods (Finite Differences, Finite Elements, IgA etc.) of PDEs leads to GLT sequences, i.e., to matrices that can be approximated by linear combinations of products involving uniform sampling diagonal matrix-sequences and Toeplitz sequences.
- A_n (from Finite Differences on a convection-diff Pb) is approximated by $D_n(a)T_n(4-2\cos(\theta_1)-2\cos(\theta_2))$ and this explains why its eigs are an approximated uniform sampling of

$$\psi(x,\theta)=a(x)(4-2\cos(\theta_1)-2\cos(\theta_2)), \quad x\in\Omega, \theta\in(-\pi,\pi)^2.$$

The GLT symbol: Toeplitz + Diagonal representation

$\mathcal{L}_a(u) = -\left(a(x)u^{'}\right)^{'}$ [rod with variable section].

$$K_n = \begin{pmatrix} d_1 & -a_2 & & \\ -a_2 & \ddots & \ddots & \\ & \ddots & \ddots & -a_n \\ & & -a_n & d_n \end{pmatrix}, \quad T_n(e^{i\theta}) = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix},$$

$$T_n(e^{-i\theta}) = T_n^T(e^{i\theta}), \quad D_n(a) = \mathrm{diag}(a(jh)), \quad h = \frac{1}{n+1}.$$

Then

$$\begin{aligned} K_n &= 2D_n(a) - D_n(a)T_n(e^{i\theta}) - T_n(e^{-i\theta})D_n(a) + E_n, \quad \|E_n\| \to 0, \\ \psi(x,\theta) &= 2a(x) - a(x)e^{i\theta} - e^{-i\theta}a(x) + 0 = a(x)(2 - 2\cos(\theta)) \end{aligned}$$

$$\{K_n\} \sim_\lambda (\psi(x,\theta), [0,1] \times (-\pi,\pi)).$$

The GLT symbol: Toeplitz + Diagonal representation

$$K_n = \Delta_n^{(2)}(a) \approx D_n(a) \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} = D_n(a) T_n(f)$$

$$\Delta_n^{(2)}(a) \approx D_n^{1/2}(a) T_n(f) D_n^{1/2}(a), \quad \psi = a(x) f(\theta).$$

$D_n(a)$ uniform diagonal sampling matrix; the decompositions (both Dyadic and Toeplitz + Diagonal) available in the multidimensional:

$$\mathcal{L}_a(u) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(a_{i,j}(x) \frac{\partial u}{\partial x_j} \right) \Rightarrow$$

$$\Delta_{\mathbf{n}}^{(2)}(a) \approx \sum_{i,j=1}^{d} D_{\mathbf{n}}(a_{i,j}) \Delta_{\mathbf{n},i,j}^{(2)}, \quad \Delta_{\mathbf{n},i,j}^{(2)} = T_{\mathbf{n}}(f_{i,j})$$

$$\psi = \sum_{i,j=1}^{d} a_{i,j}(x) f_{i,j}(\theta)$$

The GLT symbol: Toeplitz + Diagonal representation

Variable coefficients with non equispaced grid with $t_j = g(jh)$, $g([0,1]) = [0,1]$, diffeomorfism. Setting

$$\phi(a,g) = \frac{a(g)}{(g^{'})^2}$$

we have

$$-\left(a(x)u^{'}\right)^{'} \Rightarrow_{FD\ on\ \ g(jh)} \Delta_n^{(2)}(a,g) \approx \Delta_n^{(2)}(\phi(a,g)),$$

$$-\left(\phi(a,g)(x)u^{'}\right)^{'} \Rightarrow_{FD\ on\ \ jh} \Delta_n^{(2)}(\phi(a,g)) \approx D_n(\phi(a,g))T_n(f)$$

$$\psi = \phi(a,g)(x)f(\theta)$$

Using a Geometric Map the Toeplitz + Diagonal representation can be extended to the multidimensional: used in the IgA, FEM with 'graded' grids, etc.

Further Technical Insights

- The symbol can be recovered in the IgA Collocation/Galerkin setting with variable coefficient PDEs, general physical domain, general geometrical mapping.
- The symbol can be recovered in the FEM setting with variable coefficient PDEs, general physical domain, general graded griddings.
- Concerning the numerical methods, the dimensionality d is not an issue and singular mappings are not an issue.
- We are now completing the analysis when the model space is given by NURBS.

Conclusions

- In the case of constant coefficients PDEs the GLT approach and the Local Fourier Analysis lead to the same conclusions and to the same tools.
- The GLT tool has to be considered as an extension of the Local Fourier Analysis (for variable coefficients, irregular domains etc) and indeed the symbol analysis via GLT is more general and includes also integral problems, preconditioning, involved iteration matrices (PHSS), variable coefficients.
- Future work: Navier-Stokes and other vector problems to be considered, with the idea of using the spectral information and the symbol, in order to obtain faster and more robust (preconditioned) iterative solvers.

References

- GLT: Tilli LAA 98, S. LAA 03 e 06 (previous results by Kac, Parter, Widom etc)
- a.c.s: S. LAA 01, 03, 06 (previous results by Tilli)
- Spectral Tools (Bottcher, Grudky, Silbermann, Tilli, Tyrtyshnikov, Golinskii, Kuijlaars, S. + coauthors)
- FEM: Beckermann, S. SINUM 07, Garoni, S., Sesana, SIMAX 15
- IgA: Garoni, Manni, Pelosi, S., Speleers NM 14, Donatelli, Garoni, Manni, S., Speleers CMAME 14, CMAME 15, MC 16
- FDEs: Donatelli, Mazza, S. JCP 16
- Vector Problems: collaborations with Donatelli, Dorostkar, Franck, Garoni, Hughes, Manni, Mazza, Neytcheva, Ratnani, Reali, Sesana, Sonnendrücker, Speleers