Isogeometric Analysis, Symbol Approach, and Structured Matrices: from the Spectral Analysis to the Design of Fast Iterative Solvers

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Partly joint with M. Donatelli, C. Garoni, T.J.R. Hughes, C. Manni, F. Pelosi, A. Reali, D. Sesana, H. Speleers
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Model problem

\[\begin{aligned}
-\Delta u + \beta \cdot \nabla u + \gamma u &= f & \text{on } \Omega = (0,1)^d \\
u &= 0 & \text{on } \partial \Omega
\end{aligned}\]  

where \( d \geq 1, \ f \in L^2(\Omega), \ \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{R}^d, \ \gamma \geq 0 \)

Weak form

Find \( u \in H^1_0(\Omega) \) such that

\[ a(u, v) = F(v) \quad \forall v \in H^1_0(\Omega) \] 

where \( a(u, v) = \int_\Omega (\nabla u \cdot \nabla v + \beta \cdot \nabla u \ v + \gamma u \ v) \), \( F(v) = \int_\Omega f \ v \)

\( \exists ! \) solution \( u \in H^1_0(\Omega) \) to (2), called the weak solution of (1)
Choose a subspace $W \subset H_0^1(\Omega)$ with $\dim W = N < \infty$

Find $\tilde{u} \in W$ such that

$$a(\tilde{u}, v) = F(v) \quad \forall v \in W \quad (3)$$

Whatever $W$, $\exists!$ solution $\tilde{u} \in W$ to (3), which is taken as an approximation to $u$

Chosen a basis $\{\varphi_1, \ldots, \varphi_N\}$ for $W$, problem (3) is equivalent to solving the linear system

$$Au = f$$

where $A = [a(\varphi_j, \varphi_i)]_{i,j=1}^{N}$ is the **stiffness matrix** and $f = [F(\varphi_i)]_{i=1}^{N}$
In our IgA setting $W$ is chosen as a space of splines

We start with $d = 1$:

- $W = W_n^{[p]}$ = space of splines of degree $p$ on the uniform mesh $\frac{i}{n}$, $i = 0, \ldots, n$, vanishing at $x = 0, 1$
- basis of $W_n^{[p]} =$ B-spline basis

For $d \geq 2$:

$$W = W_n^{[p]} \otimes W_n^{[p]} \otimes \ldots \otimes W_n^{[p]}$$

$d$ copies

= space generated by the tensor-product B-splines vanishing on $\partial \Omega$

$A_n^{[p]} =$ stiffness matrix resulting from these choices
Asymptotic spectral distribution of the sequence of (normalized) IgA matrices

\[ \{ n^{d-2} A_n^{[p]} \}_n \]

Target: find out the symbol of the IgA matrices (Weyl sense)
Definition: spectral distribution of a sequence of matrices \( \{X_n\} \) – symbol

Let

\( \{X_n\} = \text{sequence of matrices, } X_n \text{ of size } d_n \to \infty \)
\( f : D \subset \mathbb{R}^m \to \mathbb{C} \text{ measurable function, } 0 < \text{measure}(D) < \infty \)

\( \{X_n\} \) is **distributed like \( f \) in the sense of the eigenvalues**, in symbols \( \{X_n\} \sim_{\lambda} f \), if

\[
\lim_{n \to \infty} \frac{1}{d_n} \sum_{j=1}^{d_n} F(\lambda_j(X_n)) = \frac{1}{\text{measure}(D)} \int_D F(f(x)) \, dx \quad \forall F \in C_c(\mathbb{C})
\]

\( f = \text{symbol of } \{X_n\} \)

Informal meaning of \( \{X_n\} \sim_{\lambda} f \)

**If \( f \) is smooth, then the eigenvalues of \( X_n \) behave as a uniform sampling of \( f \) over \( D \)**
Symbol of the sequence of IgA matrices \( \{n^{d-2}A_n^{[p]}\}_n \) (Szegö-type result)

\[ \phi_{[p]} = \text{cardinal B-spline} \text{ of degree } p \text{ on the uniform knot sequence } 0, 1, \ldots, p + 1 \]

For \( p \geq 1 \), consider these functions over \([-\pi, \pi]\):

\[
h_{p-1}(\theta) = \phi_{[2p-1]}(p) + 2 \sum_{k=1}^{p-1} \phi_{[2p-1]}(p-k) \cos(k\theta), \quad f_p(\theta) = (2 - 2 \cos \theta)h_{p-1}(\theta)
\]
Theorem

\[ \{n^{-1} A_n^{[p]} \}_{n} \sim \lambda \ f_p \]

The theorem holds for every \( d \) with \( d - 2 \) in place of \(-1\) and with

\[
f_p : [-\pi, \pi]^d \rightarrow \mathbb{R}
\]

\[
f_p(\theta_1, \ldots, \theta_d) = \sum_{k=1}^{d} h_p(\theta_1) \cdots h_p(\theta_{k-1}) f_p(\theta_k) h_p(\theta_{k+1}) \cdots h_p(\theta_d)
\]

\[
f_p(\theta) = (2 - 2 \cos \theta) h_{p-1}(\theta)
\]
If $d = 1$

**Theorem**

$$\{ n^{d-2} A_n[\rho] \}_n \sim \lambda f_p$$

The theorem holds for every $d$ with $d - 2$ in place of $-1$ and with

$$f_p : [-\pi, \pi]^d \rightarrow \mathbb{R}$$

$$f_p(\theta_1, \ldots, \theta_d) = \sum_{k=1}^{d} h_p(\theta_1) \cdots h_p(\theta_{k-1}) f_p(\theta_k) h_p(\theta_{k+1}) \cdots h_p(\theta_d)$$

$$f_p(\theta) = (2 - 2 \cos \theta) h_{p-1}(\theta)$$
Properties of the symbol $f_p$

For $d = 1$ the symbol is $f_p(\theta) = (2 - 2\cos \theta)h_{p-1}(\theta)$

![Figure: graph of the normalized symbol $f_p/M_{f_p}$](image)

- $\lim_{\theta \to 0} \frac{f_p(\theta)}{\theta^2} = 1$, $f_p(\theta) > 0$ for $\theta \neq 0 \Rightarrow \theta = 0$ unique zero of $f_p$ with order 2
- Setting $M_{f_p} = \max_{\theta} f_p(\theta)$, $\frac{f_p(\pi)}{M_{f_p}} \leq \frac{f_p(\pi)}{f_p(\pi/2)} = \frac{1}{2^{p-2}} \to 0$ exponentially
The normalized symbol $f_p/M_{f_p}$ has a numerical zero at $\theta = \pi$ for large $p$!

Besides the canonical zero $\theta = 0$, when $p$ is large the normalized symbol has a non-canonical numerical zero at $\theta = \pi$.

In the $d$-variate case, the situation is even worse!

Besides the canonical zero $(\theta_1, \ldots, \theta_d) = (0, \ldots, 0)$, when $p$ is large the normalized symbol has infinitely many non-canonical numerical zeros located at the $\pi$-edge points

$$\{(\theta_1, \ldots, \theta_d) : \theta_j = \pi \text{ for some } j\}$$
From the properties of the symbol:

- standard multigrid methods for the IgA matrices, which take care of the actual zero \((\theta_1, \ldots, \theta_d) = (0, \ldots, 0)\), will be optimal, i.e. with convergence rate independent of \(n\)
- for large \(p\), standard multigrid methods, which do not take care of the numerical zeros at the \(\pi\)-edge points, will have a bad convergence rate

\[\downarrow\]

multi-iterative idea (S. , Comput. Math. Appl. 1993) to be fully considered for designing optimal and robust solvers

**Target:** use **carefully the symbol** to design fast (optimal and robust) multi-iterative solvers for the IgA matrices (alternative direction by Sangalli, Tani)
Standard two-grid methods for solving linear systems with matrix $n^{d-2} A_n^{[p]}$ involve:

- a standard coarse-grid correction with **full-weighting** projector
- a post-smoothing iteration by the standard (relaxed) **Gauss-Seidel** method

**Symbol interpretation**

Full-weighting treats properly the unique zero $(\theta_1, \ldots, \theta_d) = (0, \ldots, 0)$ of the symbol... but both full-weighting and Gauss-Seidel ignore the numerical zeros arising for large $p$
... and, indeed, the standard two-grid method is optimal, but not robust...

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Table: spectral radius
Multi-iterative idea: keep the full-weighting projector for dealing with the zero \((\theta_1, \ldots, \theta_d) = (0, \ldots, 0)\) and replace the Gauss-Seidel smoother with another smoother that takes care of the numerical zeros of the symbol.
Multi-iterative methods: PCG or PGMRES as smoother

A suitable smoother is suggested by the symbol $f_p$ itself:

take as smoother the PCG or the PGMRES with preconditioner having itself a symbol $s_p$ which “deletes” the numerical zeros of our symbol $f_p$, yielding a $p$-independent preconditioned symbol $s_p^{-1} f_p$

In 1D:

$$f_p(\theta) = (2 - 2 \cos \theta) h_{p-1}(\theta)$$

$\Rightarrow$ the **preconditioned symbol** $[h_{p-1}(\theta)]^{-1} f_p(\theta) = 2 - 2 \cos \theta$ is $p$-independent
The idea can be generalized to the $d$-dimensional setting.

**For the smoother PCG / PGMRES use a preconditioner with symbol**

$$s_p(\theta_1, \ldots, \theta_d) = h_{p-1}(\theta_1) h_{p-1}(\theta_2) \cdots h_{p-1}(\theta_d)$$

**Possible choice:** Toeplitz matrix generated by $s_p$

**Remark**

Since the symbol $s_p$ of the preconditioner is a **separable trigonometric polynomial**, a linear system associated with the preconditioner is easily solvable.
Here we consider the system

\[ n^{d-2} A_n^{[p]} u = b \]

coming from the IgA approximation of (1) in the case \( d = 2, \beta = 0, \gamma = 1, f = 1 \)

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Table: number of iterations
Here we consider the system

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Here we consider the system

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coming from the IgA approximation of (1) in the case \( d = 2, \beta = 0, \gamma = 0, f = 1 \)

For the solution: V-cycle and W-cycle multigrid

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\[ n^{d-2} A_n[p] u = b \]

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For the solution: **V-cycle** and **W-cycle** multigrid

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**Table:** number of iterations

We have obtained **optimal and robust multi-iterative multigrid methods**
We have: fast multi-iterative solver for $A_n^{[p]} = \text{Parametric Laplacian matrix (PL-matrix)}$

- “Parametric”: the considered domain is the hypercube $(0,1)^d$
- “Laplacian”: the considered problem is $-\Delta u = f$

What about IgA matrices $A_n^{[p]}$ associated with full elliptic problems over general $\Omega$?

The PL-matrix $A_n^{[p]}$ is an optimal and robust CG/GMRES preconditioner for $A_n^{[p]}$
\[
\begin{aligned}
&\begin{cases}
-\nabla \cdot K \nabla u + \beta \cdot \nabla u + \gamma u = f \\
u = 0
\end{cases}
on \Omega \\
&\text{on } \partial \Omega \\
\end{aligned}
\]  \tag{4}

with $\Omega$ being a quarter of annulus:

\[-r^2 < x^2 + y^2 < R^2, \ x > 0, \ y > 0\]

and

\[
K(x, y) = \begin{bmatrix}
(2 + \cos x)(1 + y) & \cos(x + y) \sin(x + y) \\
\cos(x + y) \sin(x + y) & (2 + \sin y)(1 + x)
\end{bmatrix}
\]

\[
\beta(x, y) = \sqrt{x^2 + y^2} \begin{bmatrix}
\cos \frac{x}{\sqrt{x^2 + y^2}} \\
\sin \frac{y}{\sqrt{x^2 + y^2}}
\end{bmatrix}
\]

\[
\gamma(x, y) = xy
\]

\[
f(x, y) = x \cos y + y \sin x
\]
Isogeometric approach:

- take \( \mathbf{G} : \hat{\Omega} := (0, 1)^2 \to \Omega \) that describes \( \Omega \) exactly:

\[
\mathbf{G} : \hat{\Omega} \to \Omega \quad \mathbf{G}(\hat{x}, \hat{y}) = (x, y)
\]

\[
\begin{align*}
 x &= [r + \hat{x}(R - r)] \cos(\frac{\pi}{2} \hat{y}) \\
y &= [r + \hat{x}(R - r)] \sin(\frac{\pi}{2} \hat{y})
\end{align*}
\]

- approximate (4) with the Galerkin method:
  - approximation space: \( \mathcal{W} = (W_n^{[p]} \otimes W_n^{[p]}) \circ \mathbf{G}^{-1} \)
  - basis functions: \( \mathbf{G} \)-deformations of the tensor-product B-splines defined on \( \hat{\Omega} \)
$\mathcal{A}_n^{[p]} = \text{resulting stiffness matrix}$

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**Table:** number of iterations
Further Technical Insights

- The symbol can be recovered in the IgA Collocation/Galerkin setting with variable coefficient PDEs, general physical domain, general geometrical mapping.

- The symbol can be recovered in the FEM setting with variable coefficient PDEs, general physical domain, general graded griddings.

- Concerning the numerical methods, the dimensionality $d$ is not an issue and singular mappings are not an issue.

- We are now completing the analysis when the model space is given by NURBS.
Conclusions

- The symbol approach is useful for understanding the spectral features of the IgA matrices and for designing efficient iterative solvers.

- The symbol approach is not limited to IgA approximation techniques: it is a general tool for dealing with all local approximation techniques for PDEs, such as FD methods, FE methods, collocation methods, etc.

- As done in this presentation, we identify **two steps in the symbol approach**:
  - **find out the symbol** for the specific approximation technique under consideration and study its properties.
  - **use the symbol** and its properties to design efficient iterative solvers of Krylov, multigrid, or multi-iterative type.
Further Bibliography

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Thank you very much for your attention!
Fiorentino G., Serra S.,

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**Thank you very much for your attention**